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ON THE FREE EXPANSION OF THREE-DIMENSIONAL STREAMS OF PERFECT GAS*

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The asymptotic behavior of solutions of gasdynamics equations describing the distant regions of three-dimensional streams of inviscid and non-heat-conducting gas freely flowing into vacuum is studied. The same equations define the flows at initial sections of streams flowing into region of small, but finite, pressure or in the acceleration sections of nozzles. The principal term of coordinate expansion of inciated solutions at large distances was obtained in /1- 5/.

In this paper the expansion of solutions is constructed in the remote field in the small parameter that defines the deviation of particle velocity from the maximum possible. The second term of that expansion enables the investigation of three-dimensional flows. The obtained formulas become in cases of axisymmetric streams the expansion obtained in /6/.

Let us consider a stream of perfect gas freely flowing into vacuum along the x axis in a cylindrical system of coordinates $xr\vartheta$. Let in some cross section at $x = x_0$ of that stream the particle velocity components, total enthalpy and entropy be specified. The last two quantities are assumed constant in all cross sections normal to the x axis. The determination of parameters of the stream is required for $x > x_0$.

The stated Cauchy problem can be solved numerically, however, the coordinate expansion of the sought solution, as $x \to \infty$, is of interest. The limit value of particles velocity when $x = \infty$ is equal to the maximum possible velocity W_m.

In the case of a plane stream the problem is linearized in hodograph variables, and its solution follows from the expansion of Legendre potential in the neighborhood of point W_m. This procedure was used in /6/, where the powers of x in the coordinate expansion of plane stream parameters are indicated. In that paper they were found also for an axisymmetric problem on the assumption that $V_r/r = \partial V_r/\partial r$ in some neighborhood of the stream axis (V_r is the radial velocity component). This assumption enabled the application of the hodograph transformation for linearizing the resulting equations.

There exist another method, based on expansion in the small parameter $\varepsilon = \max(1 - W/W_m)$ of the solution /2,7/. In the first approximation it yields a stream with the velocity W_m along the x axis. Equations of the next following approximation completely agree with the equations of unsteady flow in the $r\vartheta$ plane, if in them x/W_m is substituted for time. In the case of plane flow these equations are linearized in the hodograph plane and for the Legendre potential we obtain the Darboux- Tricomi equation /8/. If the exponent of the Poisson adiabatic curve $\varkappa$ is such that the ratio $(3-\varkappa)/(\varkappa-1)$ is an even number, the general integral of the Darboux-Tricomi equation is expressed explicitly /9/. It was applied in /3/ for solving the above Cauchy problem with $\varkappa = {}^5/_3$ and ${}^7/_5$.

In the case of three-dimensional streams the general integral of unsteady flows is not known. Hence the input Cauchy problem can only be solved numerically, even in the considered here approximation. However, there is an exact solution of equations that define the plane or axisymmetric flow from a source located as a point of the x axis, taken as the coordinate origin, and is approached by the solution of the respective Cauchy problem as $x \to \infty$ /3/. That solution is, thus, the principal term of the sought coordinate expansion. It may be used for obtaining its subsequent terms.

In accordance with what was said, we shall seek the flow potential at $x \geqslant x_0$ in the form of an expansion in the small parameter ε

$$\Phi = W_m x_0 [\bar{x} + \varepsilon \Phi_1(\bar{x}, \bar{r}) + \varepsilon^2 \Phi_2(\bar{x}, \bar{r}, \vartheta) + \ldots] \qquad (1)$$
$$\bar{x} = x/x_0, \quad \bar{r} = \varepsilon^{-1/2} r/x_0$$

Henceforth we omit the bar over independent variables for the sake of simplicity.

As Φ_1 we take the source potential at the coordinate origin

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$$\Phi_1 = -\frac{1}{2m-1}(x^{2m-1}-1)+\frac{r^2}{2x},\quad 2(m-1) = -(\varkappa-1)(\nu+1) \tag{2}$$

where $\nu = 0$ is for plane and $\nu = 1$ for three-dimensional flows.

Equality (1) assumes, in fact, the existence of such x_0 for which $W/W_m = 1-\varepsilon+O(\varepsilon^2)$, $V_r/W_m = \varepsilon^{3/2}r+O(\varepsilon^{5/2})$ and $V_\vartheta/W_m = O(\varepsilon^{5/2})$. When this assumption is fulfilled, the analysis of Φ_2 shows the validity of the indicated above statement on the tendency of the Cauchy problem to the solution of the problem of source as $x\to\infty$. These relations become invalid when the stream boundary, where $W = W_m$, is approached, hence, the results obtained subsequently pertain to the stream kernel and not to its peripheral region.

The determination of the third term in formula (1) is of interest in connection with that the singularities of the three-dimensional flow at considerable distances can only be studied using Φ_2. Moreover, the analysis carried out in /6/ shows that Φ_2 contains terms whose taking into account effectively widens the region of applicability of representation (1) in comparison with its two-term analog.

To derive an equation which would satisfy $\Phi_2(x, r, \vartheta)$ we substitute formulas (1) and (2) into the equation of continuity and into the Bernoulli equation. After simple, but fairly unwieldy operations, we obtain for Φ_2 the linear inhomogeneous equation

$$\begin{aligned}&\frac{\nu+1}{2(m-1)}\left(\frac{\partial}{\partial x}+\frac{r}{x}\frac{\partial}{\partial r}\right)\left[x^{-2(m-1)}\left(\frac{\partial\Phi_2}{\partial x}+\frac{r}{x}\frac{\partial\Phi_2}{\partial r}\right)\right]+\\&\frac{\partial^2\Phi_2}{\partial r^2}+\frac{\nu}{r}\frac{\partial\Phi_2}{\partial r}+\frac{\nu}{r^2}\frac{\partial^2\Phi_2}{\partial\vartheta^2} = -\frac{\nu+1}{2}(2\varkappa+1)x^{2m-3}-\\&\frac{\nu+3}{2}\frac{r^2}{x^3}+\frac{\nu+1}{8}x^{-2m-3}r^4\end{aligned} \tag{3}$$

that has the particular solution

$$\Phi_{21} = -\frac{2\varkappa+1}{2(4m-3)}(x^{4m-3}-1)-\frac{r^4}{8x^3} \tag{4}$$

The derivatives of Φ_{21} decrease with increasing x more rapidly than the respective derivatives of Φ_1. However, for the complete analysis it is necessary to construct the general solution Φ_{20} of the homogeneous equation. We carry out in (3) the substitution of the independent variables

$$z = x^{2(m-1)},\quad \xi = (\nu+1)(\varkappa-1)^{1/2}r/x \tag{5}$$

In new variables the homogeneous equation for Φ_{20} has the simple form

$$z\frac{\partial^2\Phi_{20}}{\partial z^2}+(1-\alpha)\frac{\partial\Phi_{20}}{\partial z} = \frac{\partial^2\Phi_{20}}{\partial\xi^2}+\frac{\nu}{\xi}\frac{\partial\Phi_{20}}{\partial\xi}+\frac{\nu}{\xi^2}\frac{\partial^2\Phi_{20}}{\partial\vartheta^2},\quad \alpha = \frac{2m-1}{2m-2} \tag{6}$$

which is hyperbolic. However on plane $z = 0$ it becomes degenerate. In the physical space this plane according to (5) corresponds to infinity in the direction of the x axis, subsequently it is convenient to consider the Laplacian in the right-hand side of Eq.(6) in Cartesian coordinates

$$\xi_1 = \xi\cos\vartheta,\quad \xi_2 = \xi\sin\vartheta$$

The equation of the characteristic conoid with the apex at point $(z_0, \xi_{10}, \xi_{20})$ has the form

$$2z^{1/2}-[(\xi_1-\xi_{10})^2+(\xi_2-\xi_{20})^2]^{1/2} = 2z_0^{1/2} \tag{7}$$

It cuts out in the original plane $z = 1$ a circle with its center at point (ξ_{10}, ξ_{20}) and radius $2(1-z_0^{1/2})$.

Let us find the solution of the Cauchy problem for Eq.(6) with data in the plane $z = 1$

$$\Phi_{20} = \varphi(\xi_1, \xi_2),\ \partial\Phi_{20}/\partial z = \psi(\xi_1, \xi_2) \tag{8}$$

We use for this the Fourier method. The two functions

$$\begin{aligned}T_1 &= z^{\alpha/2}J_{|\alpha|}(2\rho z^{1/2})\exp[i(\rho_1\xi_1+\rho_2\xi_2)]\\T_2 &= z^{\alpha/2}Y_{|\alpha|}(2\rho z^{1/2})\exp[i(\rho_1\xi_1+\rho_2\xi_2)]\\\rho^2 &= \rho_1^2+\rho_2^2\end{aligned}$$

where J and Y are Bessel functions of the first and second kind, satisfy Eq.(6). Hence the sought may be represented in the form of the integral

$$\Phi_{20} = \iint\limits_{-\infty}^{+\infty} [A(\rho_1, \rho_2) T_1 + B(\rho_1, \rho_2) T_2] \, d\rho_1 \, d\rho_2 \tag{9}$$

Functions A and B are calculated using initial conditions (8), in which all quantities must be, as a prelininary, replaced by their Fourier representations. After the substitution of A and B in formula (9), it represents the solution of problem (6), (8) in the form of a fourfold integral. It is necessary to change the order of integration and calculate the internal integrals. As the result, we obtain the sought formula

$$\Phi_{20} = z^{\alpha/2} \int\limits_{(\nu-1)a}^{a} K_\nu(|\alpha|, z, \eta) \Big[Q_\nu(\xi_1, \xi_2, \eta) + \tag{10}$$

$$\frac{1}{2}(|\alpha| - \alpha) G_\nu(\xi_1, \xi_2, \eta)] \, \eta^\nu \, d\eta +$$

$$z^{(|\alpha|+\alpha)/2} \frac{\partial}{\partial z} \int\limits_{(\nu-1)a}^{a} z^{(1-|\alpha|)/2} K_\nu(|\alpha| - 1, z, \eta) \, G_\nu(\xi_1, \xi_2, \eta) \, \eta^\nu \, d\eta$$

$$a = 2(1 - z^{1/2})$$

$$K_1(\beta, z, \eta) = -\frac{1}{2} z^{-1/2} \left(\frac{\zeta}{1-\zeta}\right)^{1/2} \operatorname{ch}\left(\beta \ln \frac{\sqrt{\zeta}}{1 + \sqrt{1-\zeta}}\right)$$

$$K_0(\beta, z, \eta) = \frac{\pi^{1/2}}{2^{3/2}} \frac{1}{\sin(\beta\pi)} \left(\frac{\zeta}{z}\right)^{1/4} \Big\{ \frac{1}{2^\beta \Gamma(1+\beta) \Gamma(1/2 - \beta)} \times$$

$$\zeta^{\beta/2} F\left(\frac{1}{4} + \frac{\beta}{2}, \frac{3}{4} + \frac{\beta}{2}; 1 + \beta; \zeta\right) -$$

$$\frac{2^\beta}{\Gamma(1-\beta)\Gamma(1/2 + \beta)} \zeta^{-\beta/2} F\left(\frac{1}{4} - \frac{\beta}{2}, \frac{3}{4} - \frac{\beta}{2}; 1 - \beta; \zeta\right) \Big\}$$

$$\zeta = 4z(1 + z - \eta^2/4)^{-2}$$

where $\Gamma(a)$ denotes the gamma function and $F(a, b; c; \zeta)$ the hypergeometric Gauss function.

Functions Q_ν and G_ν are obtained from initial data (8). In the case of $\nu = 0$ we use for this a simple displacement

$$Q_0 = \psi(\xi + \eta), \quad G_0 = \varphi(\xi + \eta)$$

When the flow is three-dimensional $(\nu = 1)$, for calculating Q_1 and G_1 in addition to the displacement a supplementary averaging over the angular coordinate is required

$$Q_1 = \frac{1}{2\pi} \int\limits_0^{2\pi} \psi(\xi_1 + \eta\cos\lambda, \xi_2 + \eta\sin\lambda) \, d\lambda$$

$$G_1 = \frac{1}{2\pi} \int\limits_0^{2\pi} \varphi(\xi_1 + \eta\cos\lambda, \xi_2 + \eta\sin\lambda) \, d\lambda$$

The limits of integration in formula (10) are selected in accordance with the equation of characteristics (7) so that in the calculation of the potential Φ_{20} at point (z, ξ_1, ξ_2) only the initial data belonging to the circle in plane $z = 1$ with the center at point (ξ_1, ξ_2) and radius $2(1 - z^{1/2})$ participated.

When β approaches zero or a negative integer, the expression for the kernel $K_0(\beta, z, \eta)$ becomes indeterminate. The indeterminacy is to be opened using the standard passing to limit in each remainder of terms of hypergeometric series. Then terms with $\ln\zeta$, appear, which are the source of logarithmic dependence on x of the potential and its derivatives /6/. In the three-dimensional case for the indicated values of β, logarithms do not occur in the kernel $K_1(\beta, z, \eta)$. They, however, appear in the final calculation of Φ_{20} using formula (10).

For analyzing the behavior of $\Phi_{20}(z, \xi, \vartheta)$ as $z \to 0$ we pass in formula (10) to integration with respect to ζ, and use the representation of kernels K_1 and K_0 for small ζ. Assuming the analyticity of functions Q_ν and G_ν with respect to argument η, after simple transformations, we come to the conclusion that the derivatives of Φ_2 as $x \to \infty$, approach zero not slower than the respective derivatives of Φ_1. Note, however, that the first term in formula (2) for Φ_1 taken with an arbitrary multiplier, is a particular solution of (6). Hence it is possible to overdetermine ε so that Φ_{20} did not contain a term proportional to x^{2m-1}. Then ε changes by the quantitiy $O(\varepsilon^2)$, and the form of (1) remains unchanged. But now the derivatives of Φ_2 approach zero mor rapidly as $x \to \infty$, than the respective derivatives of Φ_1.

From this follows the validity of the indicated above principle on the asymptotics of solution of the Cauchy problem.

The coordinate expansion of velocity components on the x axis is of interest. It is obtained by the described method which yields the formulas

$$V_x/W_m = 1 - \varepsilon C_1 x^{2m-2} + \varepsilon^2 [C_2 x^{2m-3} + C_3 x^{4m-4} + O(x^{4m-5})]$$
$$V_r/W_m = \varepsilon^{3/2} (C_4 x^{2m-2} + C_5 x^{-1}) \cos\vartheta$$

When $m = 1/2$ the coefficient $\alpha = 0$ (6), and in Φ_{20} appear logarithmic terms. Accordingly the terms with coefficients C_2 and C_4 in the expression of velocity components of the x axis must be replaced by logarithmic formulas

$$C_2 x^{-2} \ln x, \; C_4 x^{-1} \ln x$$

All this conforms to results of /6/.

The coordinate expansion of Φ_{20} can be constructed using a different method by searching self-similar solutions of Eq.(6) of the form

$$\Phi_{20} = z^\gamma h_k (\xi z^{-1/2}) \cos k\vartheta$$

For h_k we obtain linear ordinary differential equations dependent on parameter γ. The parameter is determined from the condition of boundedness of derivatives of Φ_{20} when $\xi = 0$. Solution of these equations is of power form, and is not adduced here.

Let us consider the behavior of streamlines in the stream. Restricting the representation of potential (1) to the first two terms, they are straight lines $r = \psi_0 x, \vartheta = \varphi_0$, where ψ_0 and φ_0 are constant. The trinomial expansion of the potential yields the equalities.

$$r = \psi_0 x [1 + \varepsilon R(x, \xi_0, \psi_0)], \; \vartheta = \varphi_0 + \varepsilon T(x, \xi_0, \psi_0) \tag{11}$$
$$\xi_0 = (\nu + 1)(\varkappa - 1)^{1/2} \psi_0$$
$$R = \frac{1}{2(m-1)} x^{2(m-1)} + \frac{\xi_0}{2(m-1)} \int \frac{\partial \Phi_{20}}{\partial \xi}\bigg|_{\xi_0, \varphi_0} z^{-\alpha} dz$$
$$T = \frac{1}{2(m-1)\psi_0^2} \int \frac{\partial \Phi_{20}}{\partial \vartheta}\bigg|_{\xi_0, \varphi_0} z^{-\alpha} dz$$

Since R and T do not increase in absolute value as $x \to \infty$, the expansions (11) are valid in the considered here flow region. When $m = 1/2$ the second term in the formula for the coordinate r of particle trajectory approaches infinity as x is increased, i.e. the particle deviated from rectilinear trajectory $r = \psi_0 x$ at an infinitely large distance /6/. But this does not lead to an increase of the relative width of the perturbed region, hence the singularity in the asymptotic expansion does not occur. Note that in the problem of damping shock waves at large distances from bodies in a supersonic gas flow, the second term of the asymptotic expansion, while remaining considerably smaller than the principal term, leads to an infinite increase of the perturbed region relative width. This effect, called cumulative, is the cause of singularity in the asymptotic expansion that is eliminated, for instance, by the method of deformed coordinates.

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